

The Logic of Large Enough

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MPC 2010

Dan Grundy's PhD thesis (2008)

“Concepts and Calculation in Cryptography”

Available from www.cs.kent.ac.uk/~eab2/crypto/

Area with

- complex proofs (reduction, contradiction, probability);
- low confidence in proofs;
- little tradition of automation or calculation.

www.cryptofoma.org.uk

Large Enough

Asymptotic reasoning, complexity theory:
properties hold “for large enough” numbers.

$$\langle \exists X :: \langle \forall x : x > X : P \rangle \rangle$$

Assumption throughout: x and X are natural numbers.

Name this quantifier!

“Almost all”, “all but finitely many”

$$\langle \diamond x :: P \rangle \equiv \langle \exists X :: \langle \forall x : x > X : P \rangle \rangle$$

Intuition/notation: \mathbb{N} as a timeline ...

Known since 1970s, modal quantifiers since 1950s. (Emphasis: expressiveness, general class.)

Why useful?

- often left implicit in context – MPC “not done”, real risks;
- where it *is* explicit: *two* dummies;
- its negation also interesting;
- basis for more notations (eliminate more dummies).

Basic properties (unsurprising)

If x does not occur free in P then

$$\langle \langle \square x :: P \rangle \rangle \equiv P$$

Proof:

$$\langle \langle \square x :: P \rangle \rangle$$

$$\equiv \{ \text{definition of } \langle \square \rangle \}$$

$$\langle \exists X :: \langle \forall x : x > X : P \rangle \rangle$$

$$\equiv \{ \forall \text{ not trivialised: non-empty range } \}$$

$$\langle \exists X :: P \rangle$$

$$\equiv \{ \exists \text{ not trivialised: non-empty range } \}$$

P

If x does not occur free in E then

$$\langle \langle \square x :: x > E \rangle \rangle \equiv \mathbf{true}$$

From monotonicity of \forall and \exists we get:

$$\langle \forall x :: P \Rightarrow Q \rangle \Rightarrow (\langle \diamond x :: P \rangle \Rightarrow \langle \diamond x :: Q \rangle) \quad (0)$$

Intuitively

$$\langle \forall x :: P \rangle \Rightarrow \langle \diamond x :: P \rangle$$

proved using (0):

$$\langle \forall x :: P \rangle$$

$$\equiv \{ \text{left identity of } \Rightarrow, \text{ heading for (0)} \}$$

$$\langle \forall x :: \mathbf{true} \Rightarrow P \rangle$$

$$\Rightarrow \{ (0) \text{ with } P, Q := \mathbf{true}, P \}$$

$$\langle \diamond x :: \mathbf{true} \rangle \Rightarrow \langle \diamond x :: P \rangle$$

$$\equiv \{ x \text{ doesn't occur in } \mathbf{true} \}$$

$$\mathbf{true} \Rightarrow \langle \diamond x :: P \rangle$$

$$\equiv \{ \text{left identity of } \Rightarrow \}$$

$$\langle \diamond x :: P \rangle$$

Conjunction

Conjunction distributes over \diamond :

$$\langle \diamond x :: P \rangle \wedge \langle \diamond x :: Q \rangle \equiv \langle \diamond x :: P \wedge Q \rangle$$

Proof of \Leftarrow from weakening;

\Rightarrow starting with witnesses X_0 and X_1 :

$$\langle \forall x : x > X_0 : P \rangle \wedge \langle \forall x : x > X_1 : Q \rangle$$

$$\Rightarrow \{ \text{arithmetic} \}$$

$$\langle \forall x : x > X_0 \uparrow X_1 :: P \rangle \wedge \langle \forall x : x > X_0 \uparrow X_1 : Q \rangle$$

$$\equiv \{ \text{distributivity} \}$$

$$\langle \forall x : x > X_0 \uparrow X_1 :: P \wedge Q \rangle$$

$$\Rightarrow \{ \exists\text{-introduction, with } X := X_0 \uparrow X_1 \}$$

$$\langle \exists X :: \langle \forall x : x > X : P \wedge Q \rangle \rangle$$

$$\equiv \{ \text{definition of } \diamond \}$$

$$\langle \diamond x :: P \wedge Q \rangle$$

From binary to general conjunction

Weakening also accounts for

$$\langle \forall y :: \langle \diamond x :: P \rangle \rangle \Leftarrow \langle \diamond x :: \langle \forall y :: P \rangle \rangle$$

but \Rightarrow proof doesn't carry across: would need maximum of possibly infinite set. Indeed it doesn't hold: consider $x \geq y$ for P .

Consequence: can't have universally *and* large-enough quantified variables implicit in context at the same time. Worse, ...

Multiple large enough

Definition for two simultaneous dummies:

$$\langle \diamond x, y :: P \rangle \equiv \langle \exists X, Y :: \langle \forall x, y : x > X \wedge y > Y : P \rangle \rangle$$

Equivalently:

$$\langle \diamond x, y :: P \rangle \equiv \langle \exists Z :: \langle \forall x, y : x > Z \wedge y > Z : P \rangle \rangle$$

Nesting property:

$$\langle \diamond x, y :: P \rangle \Rightarrow \langle \diamond x :: \langle \diamond y :: P \rangle \rangle$$

(Proof next slide)

Proof:

$$\langle \diamond x, y :: P \rangle$$

$$\equiv \{ \text{definition of } \diamond \}$$

$$\langle \exists X, Y :: \langle \forall x, y : x > X \wedge y > Y : P \rangle \rangle$$

$$\equiv \{ \text{nesting of } \forall \text{ and of } \exists \}$$

$$\langle \exists X :: \langle \exists Y :: \langle \forall x : x > X : \langle \forall y : y > Y : P \rangle \rangle \rangle \rangle$$

$$\Rightarrow \{ \exists \forall \Rightarrow \forall \exists \}$$

$$\langle \exists X :: \langle \forall x : x > X : \langle \exists Y : \langle \forall y : y > Y : P \rangle \rangle \rangle \rangle$$

$$\equiv \{ \text{definition of } \diamond, \text{ twice} \}$$

$$\langle \diamond x :: \langle \diamond y :: P \rangle \rangle$$

\Leftarrow does not hold: $y > x$ for P . (Same counterexample stops swapping $\diamond x$ and $\diamond y$.)

Consequence: if multiple \diamond quantified variables in context, have to be explicit about their order (or lack of).

Infinitely many

The dual of \diamond :

$$\langle \square x :: P \rangle \equiv \neg \langle \diamond x :: \neg P \rangle$$

Consequently,

$$\langle \square x :: P \rangle \equiv \langle \forall X :: \langle \exists x : x > X : P \rangle \rangle$$

and also

$$\begin{aligned} \langle \diamond x :: P \rangle &\equiv \langle x : P : x \rangle \text{ is infinite} \\ \langle \square x :: P \rangle &\equiv \langle x : \neg P : x \rangle \text{ is finite} \end{aligned}$$

(Contradiction proofs of “large enough” via “infinitely many” !)

Application: Limits of sequences

$$\lim_{x \rightarrow \infty} f.x = a \equiv \langle \forall \epsilon : \epsilon > 0 : \langle \diamond x :: |f.x - a| < \epsilon \rangle \rangle$$

(one less dummy)

E.g., for fixed $c > 0$,

$$\begin{aligned} \lim_{x \rightarrow \infty} f.x = a & \\ \equiv \{ \text{definition of } \lim_{x \rightarrow \infty} \} & \\ \langle \forall \epsilon : \epsilon > 0 : \langle \diamond x :: |f.x - a| < \epsilon \rangle \rangle & \\ \equiv \{ \text{arithmetic} \} & \\ \langle \forall \epsilon : \epsilon > 0 : \langle \diamond x :: |c \cdot f.x - c \cdot a| < c \cdot \epsilon \rangle \rangle & \\ \equiv \{ \text{dummy translation: } \epsilon' := c \cdot \epsilon \} & \\ \langle \forall \epsilon' : \epsilon' > 0 : \langle \diamond x :: |c \cdot f.x - c \cdot a| < \epsilon' \rangle \rangle & \\ \equiv \{ \text{definition of } \lim_{x \rightarrow \infty} \} & \\ \lim_{x \rightarrow \infty} c \cdot f.x = c \cdot a & \end{aligned}$$

Every converging sequence is bounded.

First:

$$\lim_{x \rightarrow \infty} f.x = a$$

$$\equiv \{ \text{definition of } \lim_{x \rightarrow \infty} \}$$

$$\langle \forall \epsilon : \epsilon > 0 : \langle \diamond x :: |f.x - a| < \epsilon \rangle \rangle$$

$$\Rightarrow \{ \text{instantiation, with } \epsilon := 1 \}$$

$$\langle \diamond x :: |f.x - a| < 1 \rangle$$

$$\Rightarrow \{ \text{arithmetic} \}$$

$$\langle \diamond x :: f.x < a + 1 \rangle$$

Now we can prove boundedness of f :

$$\langle \exists b :: \langle \forall x :: f.x < b \rangle \rangle$$

$$\equiv \{ \text{range splitting on } f.x < a + 1 \}$$

$$\langle \exists b :: \langle \forall x : f.x < a + 1 \vee f.x \geq a + 1 : f.x < b \rangle \rangle$$

$$\Leftarrow \{ \text{predicate calculus} \}$$

$$\langle \exists b :: b \geq a + 1 \wedge \langle \forall x : f.x \geq a + 1 : f.x < b \rangle \rangle$$

$$\equiv \{ \uparrow \text{ exists: complement of cofinite set} \}$$

$$\langle \exists b :: b \geq a + 1 \wedge b > \langle \uparrow x : f.x \geq a + 1 : f.x \rangle \rangle$$

$$\equiv \{ \text{one point rule: take } b > \text{ both bounds} \}$$

true

Towards calculational asymptotics

Asymptotics: “at extremes things get very close” .

◊ eliminates quantification of “how extreme”
but not of “how close” (ϵ).

Asymptotic comparisons (based on absolute differences):

$$\begin{aligned} f \leftrightarrow g &\equiv \langle \forall \epsilon : \epsilon > 0 : \langle \diamond x :: |f.x - g.x| < \epsilon \rangle \rangle \\ f \triangleleft g &\equiv \langle \forall \epsilon : \epsilon > 0 : \langle \diamond x :: 0 \leq g.x - f.x < \epsilon \rangle \rangle \end{aligned}$$

Properties . . .

Asymptotic comparisons (relative differences):

$$f \prec g \quad \equiv \quad f/g \leftrightarrow 0$$

$$f \ll g \quad \equiv \quad \langle \exists C :: \langle \diamond x :: |f.x| \leq C \cdot |g.x| \rangle \rangle$$

$$f \asymp g \quad \equiv \quad f \ll g \wedge g \ll f$$

$$f \sim g \quad \equiv \quad f/g \leftrightarrow 1$$

Properties: order-like, and

$$\begin{array}{ccc}
 f \leftrightarrow g \wedge \neg(f \leftrightarrow 0) & \Rightarrow & f \sim g \\
 \sim & \cup & \asymp \\
 \asymp & \cup & \ll \\
 \asymp \circ \prec & \cup & \prec \\
 \prec \circ \asymp & \cup & \prec
 \end{array}$$

Big Oh

$$f \in O(g) \quad \equiv \quad f \ll g$$

Known properties of \ll lead to

$$f \in O(g) \wedge g \in O(h) \quad \Rightarrow \quad \begin{array}{l} f \in O(f) \\ f \in O(h) \end{array}$$

and unfold \ll for

$$\begin{array}{l} f \in O(g) \quad \Rightarrow \quad C \cdot f \in O(g) \\ f_1 \in O(g_1) \wedge f_2 \in O(g_2) \quad \Rightarrow \quad f_1 + f_2 \in O(|g_1| + |g_2|) \\ f_1 \in O(g_1) \wedge f_2 \in O(g_2) \quad \Rightarrow \quad f_1 + f_2 \in O(g_1 \uparrow g_2) \\ f_1 \in O(g_1) \wedge f_2 \in O(g_2) \quad \Rightarrow \quad f_1 \cdot f_2 \in O(g_1 \cdot g_2) \end{array}$$

$f \in O(g)$ from each asymptotic equivalence.

What next?

- expand these calculational theories; explore \limsup
- more big proofs;
- more algebraic theory for one-way functions.